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POLYNOMIAL MAPS WITH CONSTANT JACOBIAN

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ABSTRACT
It has been long conjectured that if n polynomials $f_1, \cdots, f_n$ in n variables have a (non-zero) constant Jacobian determinant then every polynomial can be expressed as a polynomial in $f_1, \cdots, f_n$. In this paper, various extra assumptions (particularly when $n = 2$) are shown to imply the conclusion. These conditions are discussed algebraically and geometrically.

1. Introduction

Let k be an algebraically closed field of characteristic zero and n an integer greater than or equal to two. Let φ be a polynomial map from k^n to k^n. That is, φ is defined by

$$\varphi : (x_1, \cdots, x_n) \mapsto (f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n))$$

where $f_1, \cdots, f_n \in k[X_1, \cdots, X_n]$ are polynomials in n variables over k. Denote the Jacobian of this map by $J(\varphi)$ or $J(f_1, \cdots, f_n)$. If φ is invertible, then its inverse is also a polynomial map and thus $J(\varphi)$ must be a non-zero constant. After suitable normalization, one can assume that $J(\varphi) = 1$.

The converse question is more difficult. That is, given a polynomial map $\varphi : k^n \to k^n$ such that $J(\varphi) = 1$, can one conclude that φ is invertible? In terms of the coordinate functions, this amounts to asking whether the polynomials $f_1, \cdots, f_n$ generate the full polynomial ring $k[X_1, \cdots, X_n]$ over k. In other words, is the k-endomorphism of $k[X_1, \cdots, X_n]$ which takes X_i to $f_i(X_1, \cdots, X_n)$ $(1 \leqq i \leqq n)$ an automorphism? The question of whether $J(\varphi) = 1$ implies that φ is invertible will be referred to as the Jacobian problem.

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In modern terminology, the nonvanishing (or, more precisely, the invertibility) of the Jacobian is equivalent to the map φ being étale. Usually one studies étale coverings — étale maps which are also finite (or proper). These are analogous to covering maps in topology. In particular, the fact that k^n is simply connected (in various topologies) guarantees that the only finite étale maps from k^n to k^n are the automorphisms. Thus an affirmative answer to the Jacobian problem requires that all étale endomorphisms of k^n be finite. There are a number of ways this type of question might admit of generalization but as far as I can determine, very little work has been done in such directions. Perhaps it is only the special nature of k^n which makes such study attractive.

Another problem which is related to the Jacobian problem is that of determining the structure of the automorphism group of the polynomial ring $k[X_1, \cdots, X_n]$. This has been studied recently by Abhyankar and Moh ([1], [2], [3]) and by Nagata [6]. In the case $n = 2$, the structure has been known for some time and goes back to Jung [5] and Van der Kulk [7]. For larger values of n, it remains an open question.

This paper deals with some conditions which, together with the fact that $J(\varphi) = 1$, guarantee that φ is an automorphism. In Section 2, the basic technique is given. Namely, one translates the question into one about a polynomial ring in one variable over a suitable field. In Section 3, this method is applied to prove that the assumption that $k(x_1, \cdots, x_n)$ is a Galois extension of $k(f_1, \cdots, f_n)$ is a sufficient additional assumption. This fact has also been proved with other techniques by Abhyankar and Heinzer (unpublished) and by Campbell [4]. In Section 4, some conditions are given which guarantee that an algebra over a field be a polynomial ring in one variable over the field. To apply the results of Section 4 to the Jacobian problem, one needs a technical assumption concerning the irreducibility of certain polynomials. This assumption is used to prove Theorem 3, which is the main result of this paper. Theorem 3 includes the statement that if $f + c = 0$ is an irreducible curve of genus 0 for all $c \in \mathbf{C}$ and if $J(f, g) = 1$, then $k[f, g] = k[x, y]$. Finally, in Section 5 there is some brief further discussion of conditions which one might hope would guarantee that certain rings are polynomial rings in one variable over a field. In light of Theorem 3, one wishes to show that if $J(f, g) = 1$ then the genus of the curves $f + c = 0$ must be 0. Proposition 9 (in Section 5) states that the genus of f cannot be equal to 1.

All rings in this article are integral domains of characteristic zero and are assumed to be contained in some universal domain. This is a technical convenience so that, for example, if A is a subring of $k[X_1, \cdots, X_n]$, and the quotient field of A is K, then $K[X_1, \cdots, X_n]$ is well-defined as the subring of

$K(X_1, \cdots, X_n)$ consisting of rational functions with denominators from A. If R is a ring, then R^* denotes the group of units of R. If R is a ring and A is a subring, then an A-derivation of R is a derivation $D: R \to R$ such that $Da = 0$ for all $a \in A$.

2.

PROPOSITION 1. *Let A be an integral domain which contains the field $\mathbf{Q}$ of rational numbers. Let K be the quotient field of A and let R be a subring of an extension field of K such that $R \cap K = A$. If there exist elements $t, u \in R$ and an A-derivation $D: R \to R$ such that $R \subset K[u]$ and $Dt = 1$, then $R = A[t]$.*

PROOF. Clearly u is transcendental over K .Now note that if $x \in R$, $y \in R$ and $xy \in A$, then $x \in A$ and $y \in A$. For it, say,

$$a_1x = F_1(u) \qquad \text{and} \qquad a_2y = F_2(u)$$

with $a_1, a_2 \in A$ and $F_1(u)$, $F_2(u) \in A[u]$, then $F_1(u)F_2(u) = a_1a_2xy \in A$. Thus $F_1(u)$, $F_2(u) \in A$, whence $x, y \in K \cap R = A$.

Next choose $a \in A$ such that $at = F(u) \in A[u]$. Then $a = aDt = F'(u)Du$. It follows that $Du \in A$ and $F'(u) \in A$. Therefore $F(u)$ is linear and so

$$at = bu + c \qquad (b, c \in A).$$

Since $a = bDu$, $ab^{-1} = Du \in A$ and $cb^{-1} = ab^{-1}t - u \in R \cap K = A$. Thus $u = ab^{-1}t - cb^{-1} \in A[t]$ and hence $R \subset K[t]$.

Finally, choose any $x \in R$ and find an element $a \in A$ such that

$$ax = b_0t^n + b_1t^{n-1} + \cdots + b_n \qquad (b_i \in A).$$

Then $aD^nx = n!\ b_0$ and thus $a \mid b_0$. It follows inductively that $a \mid b_i$ for $i = 1, \cdots, n$ and thus $x \in A[t]$.

LEMMA 1. *Let k be an algebraically closed field and let $\varphi: k^n \to k^n$ be an open dominating morphism in the Zariski topology. Then all irreducible components of $k^n - \mathrm{Im}(\varphi)$ have dimension less than or equal to $n - 2$.*

PROOF. Let $\varphi((x_1, \cdots, x_n)) = (f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n))$. If the closed set $k^n - \mathrm{Im}\,\varphi$ has a component of dimension $n - 1$, then there is a polynomial H in n variables over k such that the polynomial $H \circ \varphi = H(f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n))$ never vanishes on k^n. This implies that $H \circ \varphi$ is constant and hence that $f_1, \cdots, f_n$ are algebraically dependent over k. But this contradicts the fact that $\mathrm{Im}(\varphi)$ is dense in k^n.

COROLLARY. *Under the hypotheses of Lemma* 1,

$$k(f_1,\cdots,f_n)\cap k[X_1,\cdots,X_n]=k[f_1,\cdots,f_n].$$

PROOF. It is sufficient to show that if $P, Q \in k[X_1,\cdots,X_n]$ and if P and Q are irreducible and relatively prime, then

$$Q\circ\varphi/P\circ\varphi = Q(f_1,\cdots,f_n)/P(f_1,\cdots,f_n)\not\in k[X_1,\cdots,X_n].$$

Suppose on the contrary that $Q\circ\varphi/P\circ\varphi\in k[X_1,\cdots,X_n]$ and let

$$X=\{(x_1,\cdots,x_n)\in k^n/P(x_1,\cdots x_n)=0\}$$

and

$$Y=\{(x_1,\cdots,x_n)\in k^n/Q(x_1,\cdots,x_n)=0\}.$$

Then $X\cap\operatorname{Im}\varphi\subset Y\cap\operatorname{Im}\varphi$ and therefore, $X\cap Y\cap\operatorname{Im}\varphi = X\cap\operatorname{Im}\varphi$. Since $\dim X = n-1$, Lemma 1 implies $X\cap\operatorname{Im}\varphi\neq\emptyset$ and so $\dim(X\cap\operatorname{Im}\varphi)=n-1$. But, since P and Q are relatively prime, $\dim(X\cap Y)\leqq n-2$. This contradicts the assumption that $Q\circ\varphi/P\circ\varphi\in k[X_1,\cdots,X_n]$.

THEOREM 1. *Let k be an algebraically closed field of characteristic* 0 *and let $k[X_1,\cdots,X_n]$ be a polynomial ring in n variables over k. Let $f_1,\cdots,f_n\in k[X_1,\cdots,X_n]$ and let $\varphi: k^n\to k^n$ be the map they define. Suppose $J(\varphi)=1$. Let $K=k(f_1,\cdots,f_{n-1})$ and suppose there is a polynomial $h(X_1,\cdots,X_n)\in k[X_1,\cdots,X_n]$ such that $k[X_1,\cdots,X_n]\subset K[h]$. Then $k[f_1,\cdots,f_n]=k[X_1,\cdots,X_n]$, or, equivalently, φ is an isomorphism.*

PROOF. The map φ is étale and hence open.[†] By the Corollary to Lemma 1,

$$k[X_1,\cdots,X_n]\cap K(f_n)=k[f_1,\cdots,f_n].$$

Thus $k[X_1,\cdots,X_n]\cap K = k[f_1,\cdots,f_n]\cap K = k[f_1,\cdots,f_{n-1}]$. Define a derivation D of $K[X_1,\cdots,X_n]$ by

$$Dg = J(f_1,\cdots,f_{n-1},g).$$

Then D is plainly a K-derivation and $Df_n = 1$ Now apply Proposition 1 with $A=k[f_1,\cdots,f_{n-1}]$, $R=k[X_1,\cdots,X_n]$, $u=h$ *and* $t=f_n$ to get the conclusion.

[†] Any morphism of schemes which is flat and locally of finite type is open.

3. One application of Theorem 1 is the following result.

THEOREM 2. *Let k be an algebraically closed field of characteristic* 0 *and let* $k[X_1, \cdots, X_n]$ *be a polynomial ring in n variables over k. Let* $f_1, \cdots, f_n \in k[X_1, \cdots, X_n]$ *be polynomials such that* $J(f_1, \cdots, f_n) = 1$. *Suppose that the field* $k(X_1, \cdots, X_n)$ *is a finite Galois extension of* $k(f_1, \cdots, f_n)$. *Then* $k[f_1, \cdots, f_n] = k[X_1, \cdots, X_n]$.

Some simple facts about derivations and Galois extension will be used in the proof of Theorem 2. These are summarized in the following lemmas.

LEMMA 2. *Let R and S be integrally closed integral domains such that $S \subset R$. Suppose that the quotient field E of R is a finite Galois extension of the quotient field F of S and that $R \cap F = S$. Let D be a derivation of E such that $D(F) \subset F$ and $D(R) \subset R$. If R_0 is the integral closure of S in E, then $D(R_0) \subset R_0$.*

PROOF. Let $G = \mathrm{Gal}(E/F)$. Note that if $\sigma \in G$, then $\sigma D \sigma^{-1}$ is a derivation of E which coincides with D on F. Thus $\sigma D \sigma^{-1} = D$ and also $\sigma R_0 = R_0$ for all $\sigma \in G$. Hence $D(R_0) \subset R_0$.

LEMMA 3. *Let K be a field of characteristic* 0 *and let t be transcendental over K. Let L be a finite extension of $K(t)$ and R a subring of L which contains $K[t]$ and is finitely generated as a $K[t]$-module. Suppose that the derivation d/dt extends to a derivation D of L such that $D(R) \subset R$. Then $R = K[t]$.*

PROOF. Since K has characteristic 0, it is harmless to suppose that K is a subfield of the complex numbers. Let $x \in R$. Since R is finite over $K[t]$, x must satisfy a linear differential equation

$$D^n x + a_1(t) D^{n-1} x + \cdots + a_n(t) x = 0$$

where $a_1(t), \cdots, a_n(t) \in K[t]$. Thus, regarding x as a function of t, x must be both entire and algebraic. Thus x is a polynomial in t.

PROPOSITION 2. *Let K be a field of characteristic* 0 *and let t be transcendental over K. Let L be a finite Galois extension of $K(t)$ and let R be an integrally closed subring of L such that R has quotient field L and $R \cap K(t) = K[t]$. If there is a K-derivation $D: R \to R$ such that $Dt = 1$, then $R = K[t]$.*

PROOF. By Lemma 2, it can be assumed that R is equal to the integral closure of $K[t]$ in L. Since $K[t]$ is Noetherian and integrally closed and L is a separable extension of its quotient field, R is finite as a $K[t]$-module. Now by Lemma 3, $R = K[t]$.

PROOF OF THEOREM 2. To prove Theorem 2, let $K = k(f_1, \cdots, f_{n-1})$, $t = f_n$, $L = k(X_1, \cdots, X_n)$; $R = K[X_1, \cdots, X_n]$, and let D be the derivation of L defined by $Dg = J(f_1, \cdots, f_{n-1}, g)$. By Proposition 3, $R = K[t]$. Now apply Theorem 1 with $h = t$ to get $k[f_1, \cdots, f_n] = k[X_1, \cdots, X_n]$.

4. In trying to apply Theorem 1, one is led to the question of when a ring R containing a field K is a polynomial ring in one variable over K. There are obviously many necessary conditions which can be imposed on R and part of the problem is to determine which combinations of them provide a sufficient condition. After a bit of study it becomes apparent that the question also depends significantly on properties of the field K. In applications to the Jacobian problem, K is a pure transcendental extension of an algebraically closed field of characteristic zero (see Theorem 1). It turns out that the question is much easier if K is algebraically closed and this fact will be discussed in Section 6. Regrettably, I know of no way to use the assumption that K is purely transcendental over an algebraically closed field to any advantage. Therefore, in the present section, all results will be valid for an arbitrary field K of characteristic zero. It will be assumed, however, that the quotient field of R has transcendence degree one over K and that R is finitely generated as a K-algebra.

PROPOSITION 3. *Suppose that $R^* = K^*$ and that there is a K-derivation D of R and an element $t \in R$ such that $Dt = 1$. If x is an element of R such that $K[t, x]$ is integrally closed (in its quotient field), then $x \in K[t]$.*

PROOF. Let X be an indeterminate over $K[t]$. Then $K[t, x]$ is a quotient of $K[t, X]$ by a prime ideal. Since $K(t, x)$ has transcendence degree one over K, the prime ideal must have height one and hence be principal. Thus there is an irreducible polynomial $F(t, X)$ with coefficients in K such that

$$K[t, x] = K[t, X]/(F(t, X)).$$

Since $R^* = K^*$, K is relatively algebraically closed in R. Thus $F(t, X)$ is absolutely irreducible (i.e. it remains irreducible over all algebraic extension of K). The fact that $K[t, x]$ is integrally closed now implies that the affine curve $F(t, X) = 0$ is nonsingular. Thus the partial derivatives $F_t(t, x)$ and $F_X(t, x)$ generate the unit ideal in $K[t, x]$.

Applying D to the equation $F(t, x) = 0$ gives

$$F_t(t, x) + F_X(t, x)Dx = 0.$$

Thus, the ideal generated by $F_t(t,x)$ and $F_X(t,x)$ is generated by $F_X(t,x)$ alone. It follows that $F_X(t,x)$ is a unit of R and hence $F_X(t,x) \in K^*$. But $F(t,x)=0$ is the lowest degree equation satisfied by x over $K(t)$. Thus $F_X(t,X) \in K^*$, whence $F(t,X)$ is linear in X. Since $F(t,x)=0$, $x \in K(t) \cap R$ and, since $R^* = K^*$, this forces x to be in $K[t]$.

LEMMA 4. *If R is integrally closed, $R^* = K^*$, and the quotient field L of R has genus zero over K, then R is integral over every subring which properly contains K.*

PROOF. Suppose there are two valuation rings V_1 and V_2 of L which contain K but do not contain R. Since the genus of L over K is zero, the Riemann–Roch theorem guarantees the existence of an element $u \in L$, $u \notin K$, whose divisor involves only V_1 and V_2. In particular, both u and u^{-1} belong to all valuation rings of L over K other than V_1 and V_2. Hence u is a unit of R, or, $u \in K^*$. But this is a contradiction. It follows that there is precisely one valuation ring V of L over K which does not contain R.

Now let S be a subring of R which contains K. The integral closure S' of S in R has quotient field L and is integrally closed (since R is). Thus, either $S' = R$ or $S' = R \cap V = K$. The latter case is impossible unless $S = K$, and in the former case, R is integral over S.

PROPOSITION 4. *Let R be integrally closed, $R^* = K^*$, and suppose there is a K-derivation D of R and an element t of R such that $Dt = 1$. If the quotient field L of R has genus zero over K, then $R = K[t]$.*

PROOF. By Lemma 4, R is integral over $K[t]$ and then, by Lemma 3, $R = K[t]$.

The condition $R^* = K^*$ which appears in Propositions 3 and 4 can easily be interpreted for the particular K and R occurring in Theorem 1. Namely, if $K = k(f_1, \cdots, f_{n-1})$ and $R = K[X_1, \cdots, X_n] \subset k(X_1, \cdots, X_n)$, then $R^* = K^*$ if and only if for every irreducible polynomial P in $n-1$ variables over k, $P(f_1(X_1, \cdots, X_n), \cdots, f_{n-1}(X_1, \cdots, X_n))$ is irreducible in $k[X_1, \cdots, X_n]$. In the case $n = 2$, this condition is rephrased in the following lemma.

LEMMA 5. *Let $k[X,Y]$ be a polynomial ring in two variables over an algebraically closed field k and let $f(X,Y) \in k[X,Y]$. If $K = k(f)$ and $R = K[X,Y]$, then $R^* = K^*$ if and only if $f(X,Y) - c$ is irreducible for all $c \in k$.*

Propositions 3 and 4 can now be used to prove the following theorem.

THEOREM 3. *Let k be an algebraically closed field of characteristic 0 and let*

$f(x, y)$ and $g(x, y)$ be polynomials over k such that $f_x g_y - f_y g_x = 1$. If there is an element $a \in k$ such that $f(x, y) + ag(x, y) + b$ is irreducible for all $b \in k$, and

(a) *if there is a polynomial $h(x, y)$ such that $k[f, g, h] = k[x, y]$ or*

(b) *if the linear system of curves $f(x, y) + ag(x, y) + b = 0$ has genus 0 (for all $b \in k$)*

then $k[f, g] = k[x, y]$.

PROOF. Let $K = k(f + ag)$ and $R = K[x, y]$. By Lemma 5, $R^* = K^*$. Let D be the derivation of R, defined by $D\phi = J(f + ag, \phi)$. Then $Dg = 1$.

(a) If $k[f, g, h] = k[x, y]$, then $R = K[g, h]$. By Proposition 3, since R is integrally closed, $h \in K[g]$ and thus $R = K[g]$. Now, by Theorem 1, $k[f, g] = k[x, y]$.

(b) By hypothesis, if β is transcendental over k, the equation $f(x, y) + ag(x, y) + \beta = 0$ defines an irreducible curve of genus zero over $k(\beta)$. The affine coordinate ring $k(\beta)[x, y]/(f(x, y) + ag(x, y) + \beta)$ of this curve is isomorphic to $R = K[x, y]$. Hence the quotient field of R has genus zero over K. By Proposition 4, $R = K[g]$ and, by Theorem 1, $k[f, g] = k[x, y]$.

5. The results proved in Section 4 contain conditions which guarantee that a ring is a polynomial ring over a field. In this section some other such conditions will be discussed. The basic notation is as follows:

K is a field; L is an extension field of K of transcendence degree one over K; R is an integrally closed ring such that $K \subset R \subset L$ and its quotient field is L.

If R is to be a polynomial ring over K, the following two conditions must certainly be satisfied.

(I) $R^* = K^*$.

(II) R is a principal ideal domain.

These two conditions are far from sufficient. For example, let k be an algebraically closed field and let $k[X, Y]$ be a polynomial ring in two variables over k. Let $f(X, Y)$ be a polynomial such that $f(x, y) - c$ is irreducible for all $c \in k$. Let $K = k(f)$, $L = k(x, y)$ and $R = K[x, y]$. Then (I) and (II) are satisfied but R need not be a polynomial ring over K. Indeed R cannot be a polynomial ring over K if the curves $f(x, y) - c = 0$ have positive genus.

A third necessary condition which arises in connection with the Jacobian problem is the following:

(III) There is an element $t \in R$ and a K-derivation D of R such that $Dt = 1$.

I do not know whether the three conditions taken together are enough to guarantee that $R = K[t]$. If they are, then of course the Jacobian problem has an affirmative solution, at least in the two variable case. A more geometric formulation of these conditions sheds some light on the difficulties involved.

Let X be a complete non-singular curve which is defined over K and let S be a finite set of points of X (over the algebraic closure $\bar{K}$ of K) such that every point of X which is conjugate over K to a point of S, is also in S. The conditions (I), (II) and (III) become

(I′) There is no K-rational function on X whose divisor is supported on S. (There is no K-rational function all of whose zeros and poles are in S.)

(II′) Given any K-rational divisor D on X whose support $|D|$ is disjoint from S, there is a K-rational function f_D on X whose divisor differs from D by a divisor with support contained in S.

(III′) There is a K-rational function t on X such that the divisor of the differential dt is supported on S.

Condition (I′) can be absorbed into (II′) by adding the stipulation that f_D be unique (up to a multiplicative constant). If (I′) and (II′) hold, there is a unique differential ω whose divisor is supported on S. Then (III′) is simply the requirement that ω be the differential of a function.

Still another way to get (I′) and (II′) is to require that the group $\mathcal{D}/\mathcal{D}_l$ of K-rational divisors on X modulo linear equivalence be a finitely generated free abelian group. The prime divisors supported on S provide a free set of generators. For many fields K the Mordell–Weil theorem guarantees that this group is finitely generated. In these cases one need only check that the group is torsion-free. This is precisely the situation which occurs when K is finitely generated (as a field) over an algebraically closed field. On the other hand one has the following easy result.

PROPOSITION 7. *If K is algebraically closed, then conditions* (I), (II) *and* (III) *imply that* $R = K[t]$.

PROOF. If the genus of X is positive, then $\mathcal{D}/\mathcal{D}_l$ is isomorphic to $\mathbf{Z} \oplus J_X(K)$ where $J_X(K)$ is the group of K-rational points on the Jacobian variety of X. But if K is algebraically closed (or anywhere near it) then $J_X(K)$ is neither finitely generated nor torsion-free. Hence X must have genus zero. Now Proposition 4 guarantees that $R = K[t]$.

PROPOSITION 8. *If* (I′) *and* (III′) *are satisfied then the genus of X is not one.*

PROOF. There is a unique K-rational differential dt whose divisor is supported on S. But if the genus of X is one, there is a differential whose divisor is zero. Since dt cannot have divisor zero, the genus cannot be one.

An immediate corollary of Proposition 8 is the following:

PROPOSITION 9. *If $f(x, y)$ and $g(x, y)$ are polynomials such that $f_x g_y - f_y g_x = 1$, then the system of curves $f(x, y) + ag(x, y) + b = 0$ cannot have genus one.*

REFERENCES

1. S.S. Abhyankar and T. Moh, *Puiseaux expansion and generalized Tschirnhausen transformation I*, J. Reine Angew. Math. **260** (1973), 47–83.

2. S.S. Abhyankar and T. Moh, *Puiseaux expansion and generalized Tschirnhausen transformation II*, J. Reine Angew. Math. **261** (1973) 29–54.

3. S.S. Abhyankar and T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math. **276** (1975), 148–166.

4. L.A. Campbell, *A condition for a polynomial map to be invertible*, Math. Ann. **205** (1973), 243–248.

5. H.W.E. Jung, *Über Ganze Birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161–174.

6. M. Nagata, *On the Automorphism Group of k [x, y]*, Kinokianiya Book Store, Tokyo, 1972.

7. W. Van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. (3) **I** (1953), 33–41.

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